THE KELVIN–HELMHOLTZ INSTABILITY IN A VARIABLE MAGNETIC FIELD

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It has been established in [1], as a result of an analysis of the phenomenon of parametric instability of a tangential discontinuity in an incompressible conducting liquid arising under the action of a longitudinal magnetic field oscillating in time about an average value $\langle H \rangle \neq 0$, that a variable field stabilizes a discontinuity less effectively than does a constant field. The effect of a variable field (the case $\langle H \rangle = 0$) on the stability of an interface of uniform flows of nonmixing conducting and nonconducting liquids has been investigated in [2] with the neglect of parametric effects, and the qualitatively opposite result has been obtained: It has been erroneously claimed that a variable field always exerts a destabilizing influence. It is shown in this paper that the long-wavelength part of the spectrum of two-dimensional perturbations of an interface of conducting and nonconducting liquids is stabilized by a variable field, whereas the instability caused by the field itself is of a parametric resonance nature.

1. In a Cartesian coordinate system Oxyz with the Oz axis directed opposite to the force of gravity let the plane z = 0 be the undisturbed interface between a quiescent conducting liquid filling the region z > 0 and a heavier nonconducting liquid moving in the region z < 0 at a constant velocity $\mathbf{u} = (u_x, u_y, 0)$. We shall investigate the effect of a variable magnetic field parallel to the interface on the mechanism of the Kelvin–Helmholtz instability. In the unperturbed state the distributions of the magnetic field and the pressure are of the form

$$\mathbf{H}_1^0 = \left[H \exp\left(-\frac{z}{\delta}\right)\cos\left(\frac{z}{\delta} - \omega t\right), 0, 0\right],\ \mathbf{H}_2^0 = (\mathbf{H}\cos\omega t, 0, 0),\quad p_2^0 = -\rho_2 g z,$$
$$p_1^0 = \frac{H^2}{16\pi}\left\{1 + \cos 2\omega t - \exp\left(-\frac{2z}{\delta}\right)\left[1 + \cos\left(\frac{2z}{\delta} - 2\omega t\right)\right]\right\} - \rho_1 g z,$$

where $\delta = (2\nu_m/\omega)^{1/2}$ is the thickness of the skin layer, the subscript 1 refers to the region z > 0, and the subscript 2 refers to the region z < 0.

At some instant of time, which is taken as the origin in the following, let a vertical velocity which is small in comparison with u be imparted to a finite volume of the liquid. In the linear formulation the problem of the development of perturbations of the velocities $\mathbf{v}_1$, $\mathbf{v}_2 = \nabla U_2$, pressures p_1 and p_2, magnetic fields $\mathbf{h}_1$, $\mathbf{h}_2 = \nabla\theta$, and the interface $z = \xi(x, y, t)$ is written as follows:

$$\operatorname{div}\mathbf{v}_1 = 0,\quad \rho_1\frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}\mathbf{f},\quad \Delta U_2 = 0,\quad \Delta\theta = 0; \tag{1.1}$$

$$\frac{\partial \mathbf{h}_1}{\partial t} - \nu_m\Delta\mathbf{h}_1 = H_{1x}^0\frac{\partial \mathbf{v}_1}{\partial x} + \mathbf{G},\quad \operatorname{div}\mathbf{h}_1 = 0, \tag{1.2}$$

$$\mathbf{f} = \left[h_{1z}\frac{\partial H_{1x}^0}{\partial z},\ H_{1x}^0\left(\frac{\partial h_{1y}}{\partial x} - \frac{\partial h_{1x}}{\partial y}\right),\ H_{1x}^0\left(\frac{\partial h_{1z}}{\partial x} - \frac{\partial h_{1x}}{\partial z}\right) - h_{1x}\frac{\partial H_{1x}^0}{\partial z}\right],$$

$$\mathbf{G} = \left(v_{1z}\frac{\partial H_{1x}^0}{\partial z}, 0, 0\right),\quad p_2 = -\rho_2\left(\frac{\partial U_2}{\partial t} + \mathbf{u}\nabla U_2\right);$$

$$z = 0\colon \frac{\partial \xi}{\partial t} = v_{1z},\quad \frac{\partial \xi}{\partial t} + u_x\frac{\partial \xi}{\partial x} + u_y\frac{\partial \xi}{\partial y} = \frac{\partial U_2}{\partial z}; \tag{1.3}$$

$$z = 0\colon p_1 - p_2 = \xi\left\{g(\rho_1 - \rho_2) - \frac{H^2}{8\pi\delta}\left[1 + \sqrt{2}\cos\left(2\omega t - \frac{\pi}{4}\right)\right]\right\} + \alpha\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right); \tag{1.4}$$

$$z = 0\colon h_{1x} = \frac{\partial\theta}{\partial x} + \frac{\sqrt{2}\,\xi}{\delta}H\cos\left(\omega t - \frac{\pi}{4}\right),\quad h_{1y} = \frac{\partial\theta}{\partial y},\quad h_{1z} = \frac{\partial\theta}{\partial z}; \tag{1.5}$$

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$$z \to +\infty: \mathbf{v}_1 \to 0,\ \mathbf{h}_1 \to 0; \quad z \to -\infty: U_2 \to 0,\ \theta \to 0; \tag{1.6}$$

$$t = 0: \xi = 0,\ \mathbf{h}_1 = 0,\ v_{jz} = V_j,\ j = 1, 2; \quad V_1(x, y, 0) = V_2(x, y, 0). \tag{1.7}$$

Here α is the surface tension coefficient; it is assumed in the writing of (1.5) that the ratio ξ/δ is of the first order of smallness. Taking account in (1.7) of the initial perturbations of the interface and the magnetic field only complicates somewhat the calculations, but does not result in a qualitative change in the results.

Let F be the Fourier transformation operator in the coordinates x and y, $\mathbf{k} = (k_x, k_y, 0)$ be the wave vector, L be the Laplace transformation operator in the time t, and s be the parameter of this transformation. Let us introduce the notation

$$F\xi = \eta,\ l = \eta \cos(\omega t - \pi/4),\ F\theta = \varphi,\ F\mathbf{v}_1 = \mathbf{w},\ F\mathbf{h}_1 = \mathbf{b},$$
$$Fp_1 = q,\ FU_2 = P,\ FV_1(x, y, 0) = V,\ Ll = M,\ L\varphi = \Phi,\ L\mathbf{b} = \mathbf{B}.$$

2. It is not difficult to see that with $V_j = V_j(y, z)$ the conducting fluid does not perturb the magnetic field in the lower half-space. In this case the problem (1.1)-(1.7) has a solution of the form

$$\xi = \xi(y, t),\quad \mathbf{v}_1 = \nabla U_1(y, z, t),\quad p_1 = -\rho_1 \frac{\partial U_1}{\partial t} - \frac{1}{4\pi} \mathbf{H}_1^0 \mathbf{h}_1,\quad \mathbf{h}_1 = [h_{1x}(y, z, t),\ 0, 0],\quad U_2 = U_2(y, z, t),\quad \theta = 0.$$

After simple calculations we arrive at the following problem, which describes the development of the Fourier components of the perturbation of the interface:

$$(\rho_1 + \rho_2)\frac{d^2\eta}{dt^2} - 2i\rho_2 k_y u_y \frac{d\eta}{dt} + \left[k_y g(\rho_2 - \rho_1) - \rho_2 (k_y u_y)^2 + \alpha k_y^3\right]\eta = 0,$$
$$i = \sqrt{-1};\quad t = 0:\ \eta = 0,\quad \frac{d\eta}{dt} = V.$$

The well-known criterion of the stability of a discontinuity of the tangential velocity $\mathbf{u} = (0, u_y, 0)$ in the presence of the forces of gravity and surface tension [3] follows from the condition of boundedness of $|\eta|$. Thus, plane perturbations of an interface whose crests are parallel to the unperturbed magnetic field are not acted on by the field.

3. It is evident from the condition of conjugacy of h_{1x} and h_{2x} on the interface (1.5) that a dependence of $\mathbf{h}_1$ on time is exhibited both due to the oscillation of the unperturbed field with frequency ω and due to vibrations of the conducting liquid at hydrodynamic frequencies Ω. The estimates show that when $\Omega/\omega \ll 1$, one can neglect the terms which take account of the effect of the hydrodynamical perturbations on $\mathbf{h}_1$ appearing on the right-hand side of the linearized induction equation (1.2). Changing over in the problem (1.1)-(1.7) to the transforms, we obtain, as a result of uncomplicated calculations,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 w_z}{\partial z^2} - k^2 w_z\right) = \frac{ik_x}{4\pi\rho_1}\left(b_z \frac{\partial^2 H_{1x}^0}{\partial z^2} - \frac{1}{\nu_m} H_{1x}^0 \frac{\partial b_z}{\partial t}\right),\quad k^2 = k_x^2 + k_y^2; \tag{3.1}$$

$$\frac{d^2\mathbf{B}}{dz^2} - k^2\left(\frac{s}{m} + 1\right)\mathbf{B} = 0,\quad \frac{dB_z}{dz} - i\mathbf{k}\mathbf{B} = 0,\quad m = \nu_m k^2; \tag{3.2}$$

$$\frac{d^2\Phi}{dz^2} - k^2\Phi = 0,\quad P = \frac{e^{kz}}{k}\left(\frac{d\eta}{dt} - i\mathbf{k}\mathbf{u}\eta\right); \tag{3.3}$$

$$q = \frac{1}{k^2}\left\{\frac{1}{4\pi}\left[ik_x b_z \frac{\partial H_{1x}^0}{\partial z} - k_y H_{1x}^0 (k_y b_x - k_x b_y)\right] - \rho_1 \frac{\partial^2 w_z}{\partial t \partial z}\right\}; \tag{3.4}$$

$$z = 0:\ \frac{\partial w_z}{\partial t} = \frac{d^2\eta}{dt^2},\quad b_x = \frac{\sqrt{2}}{\delta} Hl - ik_x\varphi,\quad b_y = -ik_y\varphi,\quad b_z = \frac{\partial\varphi}{\partial z}; \tag{3.5}$$

$$z = 0:\ q + \rho_2\left(\frac{\partial P}{\partial t} - i\mathbf{k}\mathbf{u}P\right) = \eta\left\{g(\rho_1 - \rho_2) - \frac{H^2}{8\pi\delta}\left[1 + \sqrt{2}\cos\left(2\omega t - \frac{\pi}{4}\right)\right] - \alpha k^2\right\}; \tag{3.6}$$

$$z \to +\infty:\ \frac{\partial w_z}{\partial t} \to 0,\ \mathbf{B} \to 0;\quad z \to -\infty:\ \Phi \to 0; \tag{3.7}$$

$$t = 0:\ \eta = 0,\quad \frac{d\eta}{dt} = V. \tag{3.8}$$

Having written out the solutions of the boundary-value problems (3.2), (3.3), (3.5), and (3.7),

$$B_x = \frac{\sqrt{2}}{\delta} H\,(M e^{-\zeta kz} - \beta^2 N), \quad B_y = -\frac{\sqrt{2}\,\beta\gamma}{\delta} HN,$$
$$B_z = -\frac{i\sqrt{2}\,\beta}{\delta} HN, \quad \Phi = -\frac{i\sqrt{2}\,\beta}{k\delta}\,\frac{e^{kz}}{\zeta+1} HM,$$
$$N = \frac{e^{-\zeta kz}}{\zeta+1} M, \quad \zeta = \sqrt{\frac{s}{m}+1}, \quad \beta = \frac{k_x}{k}, \quad \gamma = \frac{k_y}{k}$$

and performed an inverse Laplace transformation, we find

$$b_x = \frac{\sqrt{2}}{\delta} H \left\{ \int_0^t e^{-m\tau}\,\mathrm{Erf}\left(\frac{kz}{2\sqrt{m\tau}}\right)\left[\frac{dl\,(t-\tau)}{dt} + ml\,(t-\tau)\right] d\tau - \beta^2 \chi(z,t) \right\}, \quad b_y = -\frac{\sqrt{2}\,\beta\gamma}{\delta} H\chi(z,t), \tag{3.9}$$
$$b_z = -\frac{i\sqrt{2}\,\beta}{\delta} H\chi(z,t), \quad \varphi = -\frac{i\sqrt{2}\,\beta}{k\delta} H\chi(0,t)\,e^{kz},$$
$$\chi(z,t) = \int_0^t l\,(t-\tau)\left\{\sqrt{\frac{m}{\pi\tau}}\exp\left[-m\tau - \frac{(kz)^2}{4m\tau}\right] + m e^{kz}\,\mathrm{Erf}\left(\frac{kz+2m\tau}{2\sqrt{m\tau}}\right)\right\} d\tau, \quad \mathrm{Erf}\,z = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,dx.$$

Bearing (3.9) in mind, one can write the solution of the boundary-value problem (3.1), (3.5), and (3.7) in $\partial w_z/\partial t$ and calculate the derivative $\partial^2 w_z/\partial t\partial z$. Next, having substituted (3.4) into the dynamical condition on the interface (3.6), we obtain an equation which describes the development of the Fourier components of the perturbation of the interface. Introducing the dimensionless variables $t_* = \omega t$ and $\tau_* = \omega\tau$, we have (the asterisks are omitted)

$$\frac{d^2\eta}{dt^2} - 2i\varepsilon_1\frac{d\eta}{dt} + \left[\varepsilon_0 + \varepsilon_2\cos\left(2t - \frac{\pi}{4}\right)\right]\eta = \varepsilon_3\int_0^t K(t,\tau)\,\eta(t-\tau)\,d\tau, \tag{3.10}$$
$$K(t,\tau) = \left(\frac{e^{-\varkappa\tau}}{\sqrt{\pi\varkappa\tau}} - \mathrm{Erf}\sqrt{\varkappa\tau}\right)[\cos(2t-\tau) - \sin\tau],$$
$$\varepsilon_0 = \left(\frac{\Omega}{\omega}\right)^2, \quad \varepsilon_1 = \frac{\sqrt{2\varkappa}}{\delta}\frac{\lambda r_2}{\omega}, \quad \varepsilon_2 = \sqrt{\varkappa}\,\Gamma, \quad \varepsilon_3 = \varkappa\Gamma, \quad r_j = \frac{\rho_j}{\rho_1+\rho_2}, \quad j = 1, 2,$$
$$\Omega^2 = g\,(r_2 - r_1)\frac{\sqrt{2\varkappa}}{\delta} + \omega^2\Gamma\sqrt{\frac{\varkappa}{2}} - 2\varkappa r_2\left(\frac{\lambda}{\delta}\right)^2 + \left(\frac{\sqrt{2\varkappa}}{\delta}\right)^3\frac{\alpha}{\rho_1+\rho_2},$$
$$\varkappa = \frac{(k\delta)^2}{2}, \quad \lambda = \beta u_x + \gamma u_y, \quad \Gamma = \frac{1}{4\pi(\rho_1+\rho_2)}\left(\frac{\beta H}{\omega\delta}\right)^2.$$

Upon switching to the variable t_*, the initial conditions (3.8) are transformed in an obvious fashion. We shall denote $\varepsilon = \max\,(|\sqrt{\varepsilon_0}|, \varepsilon_j)$, $c_j = \varepsilon_j/\varepsilon$, $j = 1, 2, 3$, $c_0 = \varepsilon_0/\varepsilon$. Let us rewrite the problem (3.10) and (3.8) in the form

$$\frac{d\eta}{dt} = \sqrt{\varepsilon}\,\vartheta, \quad \frac{d\vartheta}{dt} = \sqrt{\varepsilon}\left\{2i\sqrt{\varepsilon}\,c_1\vartheta - \left[c_0 + c_2\cos\left(2t - \frac{\pi}{4}\right)\right]\eta \right.$$
$$\left. + c_3\int_0^t K(t,\tau)\,\eta(t-\tau)\,d\tau\right\}; \quad t = 0: \eta = 0, \quad \vartheta = \frac{\omega}{\sqrt{\varepsilon}}V. \tag{3.11}$$

Equation (3.10) is obtained in the approximation $|\sqrt{\varepsilon_0}| \ll 1$. Assuming also $\sqrt{\varepsilon} \ll 1$ and averaging (3.11) by the second scheme proposed in [4], we obtain the Cauchy problem for a system of two differential equations. Proceeding in the average problem to a single second-order equation, we have

$$\frac{d^2\mu}{dt^2} - 2i\varepsilon_1\frac{d\mu}{dt} + \left[\varepsilon_0 + \Gamma\frac{\sqrt{\varkappa}}{2}(1+\Lambda)\right]\mu = 0; \quad t = 0: \mu = 0, \quad \frac{d\mu}{dt} = \omega V,$$
$$\Lambda(\varkappa) = \left(\frac{1}{\sqrt{\varkappa^2+1}} - \frac{\varkappa}{\varkappa^2+1}\right)^{\frac{1}{2}} + \varkappa\left(\frac{1}{\sqrt{\varkappa^2+1}} + \frac{\varkappa}{\varkappa^2+1}\right)^{\frac{1}{2}} - \sqrt{2\varkappa}.$$

It follows from this that with the specified parameters of the unperturbed flow those harmonics $\eta(k_x, k_y, t)$ are stable for which the condition

$$(\beta u_x + \gamma u_y)^2 < \frac{\rho_1+\rho_2}{\rho_1\rho_2}\left\{\frac{1}{\sqrt{2\varkappa}}\left[g\delta(\rho_2-\rho_1) + \frac{\beta^2}{8\pi}H^2(1+\Lambda)\right] + \frac{\alpha}{\delta}\sqrt{2\varkappa}\right\} \tag{3.12}$$

is satisfied. On the positive half-axis the function $\Lambda(\varkappa) > 0$, so that, with $k_x \neq 0$, the variable magnetic field exerts a stabilizing effect; the field, just as the force of gravity, stabilizes only the long-wavelength part of the spectrum. In the case in which the discontinuity of the velocity $\mathbf{u} = (u_x, 0, 0)$ is parallel to the lines of force of the unperturbed field, one can obtain from (3.12) a sufficient condition of stability for $\Omega/\omega \ll 1$

$$u^4 < 4\alpha\left[g(\rho_2-\rho_1) + \frac{H^2}{8\pi\delta}\right]\left(\frac{\rho_1+\rho_2}{\rho_1\rho_2}\right)^2.$$

4. Let us consider the limiting case in which the thickness of the skin layer and the amplitude of the wave (which is small in comparison with its length) are of the identical order of magnitude. In this situation the perturbation of the magnetic field caused by distortion of the interface is comparable in order of magnitude with the unperturbed field, due to which the linearized induction equation (1.2) is inapplicable. The problem under discussion is thereby simplified, since to the assumed degree of accuracy one can replace the skin layer with a surface of discontinuity of the tangential component of the magnetic field, on which the surface ponderomotive force is localized [5]. In this approximation the distributions of the field and the pressure in the unperturbed state are described by the expressions

$$\mathbf{H}_1^0 = 0, \quad p_1^0 = -\rho_1 gz + \frac{H^2}{8\pi}\cos^2\omega t,$$
$$\mathbf{H}_2^0 = (H\cos\omega t, 0, 0), \quad p_2^0 = -\rho_2 gz.$$

In this case, one should set $\mathbf{h}_1 = 0$ in the problem (1.1)-(1.7), and in place of the conditions (1.5), which are obtained within the framework of the assumption of continuity of the field on the interface, it is necessary to formulate for the potential θ a condition which expresses the continuity of only the normal component of the field, and, in addition, to take account in the dynamical condition (1.4) of the perturbation of the surface ponderomotive force

$$\frac{\partial\theta}{\partial x} = H\frac{\partial\xi}{\partial x}\cos\omega t,$$
$$p_1 - p_2 = \xi g(\rho_1-\rho_2) + \frac{H}{4\pi}\frac{\partial\theta}{\partial x}\cos\omega t + \alpha\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right).$$

With these changes in the formulation of the problem (1.1)-(1.7) taken into account, the equation for $\eta(k_x, k_y, t)$ takes the form

$$(\rho_1+\rho_2)\frac{d^2\eta}{dt^2} - 2i\rho_2\mathbf{ku}\frac{d\eta}{dt} + \left[gk(\rho_2-\rho_1) - \rho_2(\mathbf{ku})^2 + \alpha k^3 + \frac{1}{8\pi}(k_x H)^2(1+\cos 2\omega t)\right]\eta = 0.$$

This equation is easily reduced to the standard form of a Mathieu equation. The stability diagram of the solutions of the Mathieu equation [6] confirms the qualitative conclusion drawn for $\Omega/\omega \ll 1$ about the influence of the field on the development of harmonics. In the general case, when the ratio Ω/ω is not small, it follows from the stability diagram that when $k_x \neq 0$ one can stabilize any harmonic which is unstable without a field by the choice of the amplitude H. The effect of the field is of a twofold nature: Stabilizing some regions of the spectrum, it moreover causes a parametric instability of the harmonics corresponding to other regions of the spectrum.

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